Math 7222 Lecture 28 - Katok Entropy Rigidity

[Entropy for flow $\mathcal{F} = (f_t)$
$h(\mu, \mathcal{F}) := h(\mu, f_1)$ makes sense is because Abramov proved $h(\mu, f_t) = |t| h(\mu, f_1)$]

(M, g) compact Riemannian surface with negative curvature

$h^*(g)$ - growth rate of periodic orbits (of course we proved $h^*(g) = h_{top}(g)$)

(f_t^g) geodesic flow w.r.t. the metric g

μ_L^g - Liouville measure for g (on S^gM)
unit tangent bundle w.r.t. g

V^g = total area of M (volume)

m^g = normalized Riemannian volume on M
to be a prob. measure
i.e. $m^g = \frac{Vol^g}{V^g}$ on M

Note for constant -ve curvature K
Euler characteristic

$$h(\mu_L) = \sqrt{-K} = \left(\frac{-2\pi \chi(M)}{V^g} \right)^{1/2}$$

<u>Theorem</u> (Katok '82)

For any neg. curv. g on M

$$h(\mu_L) \leq \left(\frac{-2\pi \chi(M)}{V^g} \right)^{1/2}$$

with equality iff <u>constant</u> neg. curv.

[i.e. fixing area: $g \to h(\mu_L^g)$ has its maximum at constant neg. curvature]

Geometry fact:

Any negative curvature metric g on a surface can be written $g = \rho g_0$ where $\rho > 0$ is a scalar function on M and g_0 is a constant neg curv. metric with $V^g = V^{g_0}$

Comparing metrics

g_1, g_2 Riem. metrics

" average value of $\frac{g_2\text{-length}}{g_1\text{-length}}$ "

Definition

$$[g_1 ; g_2] = \int_{v \in S^{g_1}M} \|v\|_{g_2} \, d\mu_L^{g_1}(v)$$

Lemma 1 $\quad [g_1 ; \rho g_1] = \int_M \rho^{1/2} \, dm^{g_1}$

[Recall: locally $\mu_L^g = m^g \times (\text{Haar on } S^1)$]

Proof idea:

Note that $\|v\| = \sqrt{g(v,v)}$

so $\|v\|_{\rho g_1} = \sqrt{\rho(\pi v)\, g_1(v,v)}$

$= \sqrt{\rho(\pi v)}\, \|v\|_{g_1}$ $\blacksquare$

[Recall footpoint map $\pi : SM \to M$, $(x,v) \to x$]

Details: Exercise

Lemma 2 If $g_2 = \rho g_1$ and g_1, g_2 have same volume

a) $[g_1 ; g_2] = [g_2 ; g_1]$

b) $\int_M \rho^{1/2} \, dm^{g_1} \leq 1$ with equality if and only if $\rho \equiv 1$

$\overset{\shortparallel}{[g_1 ; g_2]}$ by Lemma 1

[If $\dim M = n$ get $\rho^{n/2}$]

Proof a) If $g_2 = \rho g_1$, then $\mathrm{Vol}_{g_2} = \rho \, \mathrm{Vol}_{g_1}$

so $dm^{g_2} = \frac{V^{g_1}}{V^{g_2}} \rho \, dm^{g_1}$

So if $v^{g_2} = v^{g_1}$, then $[g_1; g_2] = \int_M \rho^{1/2} \, dm^{g_1}$

$= \int_M \rho^{-1/2} \, dm^{g_2}$

$= [g_2; g_1]$

b) $\int_M \rho \, dm^{g_1} = \int_M 1 \, dm^{g_2} = 1$

↑ since m is prob. measure

So by Cauchy-Schwartz (or Jenssen), $\left(\int_M \rho^{1/2} dm^{g_1}\right)^2 \leq 1$

with equality iff $\rho = 1$ ∎

Combining Lemma 1 and 2

$$\boxed{[g_1; g_2] = [g_2; g_1] \leq 1 \quad \text{with equality iff } \rho = 1}$$

<u>Key result</u>

If $g_2 = \rho g_1$, both with neg. curv. Then

$$\boxed{h^*(g_2) \geq h(\mu_L^{g_1}) \big/ [g_1; g_2]}$$

[Equality is free when $\rho = 1$ (constant neg. curvature)]

<u>Proof</u> of Katok's thm given Key result:

Let $g_0 = \rho g_1$ with same volume V
and g_0 has constant neg. curv.

Then $\left(-\frac{2\pi \chi(M)}{V}\right)^{1/2} = h_{top}(g_0) = h^*(g_0)$

Key result $\rightarrow \geq h(\mu_L^{g_1}) \big/ [g_1; g_0]$

By lemma 2. $[g_0; g_1] = [g_1; g_0] \leq 1$

Thus $h(\mu_L^{g_1}) \leq \left(\frac{-2\pi\chi(M)}{V}\right)^{1/2} [g_1; g_0]$

$$\leq \left(\frac{-2\pi\chi(M)}{V}\right)^{1/2}$$

Equality holds only if $[g_1; g_0] = 1 \iff \rho = 1$ ∎

<u>Proof of key result</u>

For $T > 0$, $\varepsilon > 0$, let

$$A_{\varepsilon, T} = \left\{ v \in S^{g_1}M : \left| \frac{1}{T}\int_0^T \| f_t^{g_1} v \|_{g_2} \, dt - [g_1; g_2] \right| < \varepsilon \right\}$$

We know that $\mu_L^{g_1}$ is ergodic (Hopf argument last time)

By ergodic theorem

$$\mu_L^{g_1}(A_{\varepsilon, T}) \to 1 \quad \text{as } T \to \infty$$

[Recall $[g_1, g_2] = \int_{SM} \|v\|_{g_2} \, d\mu_L^{g_1}$. So we ergodic theorem on observable $v \mapsto \|v\|_{g_2}$]

To be continued next time...

Math 7222 Lecture 29 - Katok entropy (ctd.)

Last time: Proved $g \to h(\mu_L^g)$ is maximized at constant neg. curv. surface given the key result.

Key result

If $g_2 = \rho g_1$ both with neg. curv. Then

$$\boxed{h^*(g_2) \geq \frac{h(\mu_L^{g_1})}{[g_1; g_2]}}$$

TODAY: PROOF OF KEY RESULT

For $T > 0$, $\varepsilon > 0$, let

$$A_{\varepsilon, T} = \left\{ v \in S^{g_1}M : \left| \frac{1}{T}\int_0^T \| f_t^{g_1} v\|_{g_2}\, dt - [g_1;g_2] \right| < \varepsilon \right\}$$

and $A_{\varepsilon, \geq T} = \bigcap A_{\varepsilon, T}$

We know that $\mu_L^{g_1}$ is ergodic (Hopf argument last time)

By ergodic theorem

$$\mu_L^{g_1}(A_{\varepsilon, \geq T}) \to 1 \quad \text{as } T \to \infty$$

[Recall $[g_1, g_2] = \int_{SM} \|v\|_{g_2}\, d\mu_L^{g_1}$. So use ergodic theorem on observable $v \to \|v\|_{g_2}$]

Let $B_{\varepsilon, T} = \{ v \in S^{g_1}M : \exists t$ with $T \leq t \leq (1+\varepsilon)T$ with $d_{S^{g_1}M}(v, f_t^{g_1} v) < \varepsilon \}$

and $B_{\varepsilon, \geq T_0} = \bigcap_{T = T_0}^{\infty} B_{\varepsilon, T}$

<u>Claim</u> $\mu_L^{g_1}(B_{\varepsilon, T}) \to 1$ as $T \to \infty$

Let ξ be finite partition of $S^{g_1}M$ into sets of diam $< \varepsilon$. Let $c \in \xi$

By ergodicity, $\exists Z$ with full measure so for all $v \in Z$, $\lim \frac{1}{T} S_T \chi(v) = \mu_L^{g_1}(c)$

where $S_T \ell = \int_0^T \ell(f_t v)\,dt$ and $\chi = \chi_C$

Subclaim: given $v \in Z$, we must have $T_0 = T_0(v)$ so for all $T \geq T_0$ $S_T \chi(v) < S_{(1+\varepsilon)T} \chi(v)$

Proof of subclaim: If not, $\exists\, T_k \to \infty$ with

$$S_{T_k} \chi(v) = S_{T_k(1+\varepsilon)} \chi(v)$$

Thus $\frac{1}{1+\varepsilon} \frac{1}{T_k} S_{T_k} \chi(v) = \frac{1}{(1+\varepsilon)T_k} S_{T_k(1+\varepsilon)} \chi(v)$

Taking limits $\frac{1}{1+\varepsilon} \mu(C) = \mu(C)$. Contradiction $\square$

It follows that $f_t v \in C$ for some $t \in [T, (1+\varepsilon)T]$ (for any $T \geq T_0$)

Thus for a.e. $v \in C$, $\exists\, T_0 = T_0(v)$ so $v \in B_{\varepsilon, \geq T_0}$

Carry out same argument for each C.

Thus $\mu_L\left(\bigcup_T B_{\varepsilon, \geq T} \right) = 1$

Since $B_{\varepsilon, \geq T}$ increases with T

So $\mu_L\left(\bigcup_T B_{\varepsilon, \geq T} \right) \to 1$

In particular, $\mu_L\left(\bigcup_T B_{\varepsilon, T} \right) \to 1$ $\blacksquare$

We conclude that $\forall\, \varepsilon, \delta > 0$ small, $\exists\, T$ with $\mu_L^{g_1}\left(A_{\varepsilon, \geq T} \cap B_{\delta, \geq T} \right) > 1/2$

Fix $\varepsilon > 0$. Let $\delta > 0$ be such that if $d(v, f_t v) < \delta$ $\exists$ closed geodesic γ_v of period t' with $|t-t'| < 1$ such that $d_t(v, w) < \varepsilon$, where $w = \dot{\gamma}_v(0)$ (i.e. δ constant coming from Anosov closing lemma to closing scale ε)

Let T be large enough that

$$\mu_L^{g_1}\left(A_{\varepsilon, \geq T} \cap B_{\delta, \geq T} \right) > \tfrac{1}{2}$$

Let E_t be maximal $(t, 3\varepsilon)$-sep. set for $(A_{\varepsilon, \geq T} \cap B_{\delta, \geq T})$

Recall that $\frac{1}{t} \log \# E_t \to h(\mu_L^{g_1})$ by Katok entropy formula for flows

For each $v \in E_t$, we obtain a closed geodesic γ_v from the Anosov closing lemma

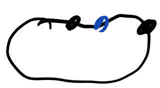
Let $w(v)$ be on the geodesic γ_v with $d_t(v, w(v)) < \varepsilon$

Then $\{ w(v) : v \in E_t \}$ is (t, ε)-separated

min. distance betw $w(v)$'s

and $\# \{ \gamma_v : v \in E_t \} \geq \dfrac{\varepsilon}{t(1+\varepsilon)+\varepsilon} \# E_t$

upper bound on period of γ



Since (M, g_1) has negative curvature all these geodesics belong to different free homotopy classes

Lemma $\exists K(\varepsilon)$ (ind. of t and satisfies $K(\varepsilon) \to 0$ as $\varepsilon \to 0$) so that the g_2-length of γ_v satisfies

$$L_{g_2}(\gamma_v) \leq t\left([g_1, g_2] + K(\varepsilon) \right)$$

[γ_v is closed geodesic in g_1. But w.r.t. g_2 it is a closed curve (not geodesic)]

Proof

Estimate g_2-length of γ_v:

[here $t' < (1+\varepsilon)t + \varepsilon$ is a period of γ_v, so least period is some t'/n]

$$L_{g_2}(\gamma_v) = \int_0^{t'/n} \| f_s^{g_1} w(v) \|_{g_2} \, ds$$

$$\leq \int_0^{t'} \| f_s^{g_1} w(v) \|_{g_2} \, ds$$

$$= \int_0^{t} \| f_s^{g_1} w(v) \|_{g_2} \, ds + \int_t^{t'} \| f_s^{g_1} w(v) \|_{g_2} \, ds$$

$$\leq \int_0^t \| f_s^{g_1} v \|_{g_2} \, ds + \left(\int_0^t [\| f_s^{g_1} w(v) \| - \| f_s^{g_1} v \|] \, ds \right) + \varepsilon t \underset{\max \|\cdot\|_{g_2}}{\underset{\shortparallel}{K_0}}$$

Let $Var(\varepsilon) = \max (| \|v\| - \|w\| | : d_{S^{g_1} M}(v, w) < \varepsilon)$

and recall that $\int_0^t \| f_s^{g_1} v \|_{g_2} \, ds < t([g_1, g_2] + \varepsilon)$

We obtain $L_{g_2}(\gamma_v) \leq t([g_1, g_2] + \varepsilon + Var(\varepsilon) + \varepsilon K_0)$

Letting $K(\varepsilon) = \varepsilon + Var(\varepsilon) + \varepsilon K_0$, we're done ∎

Let Π be free homotopy class containing γ_v

$\exists$ a shortest curve w.r.t. g_2, denote α, which is a closed geodesic w.r.t. g_2

We have $L_{g_2}(\alpha) \leq L_{g_2}(\gamma_v) \leq t([g_1, g_2] + K(\varepsilon))$

Thus

$$\# Per^{g_2}(t([g_1, g_2] + K(\varepsilon))) \geq \# \{ \gamma_v : v \in E_t \}$$

$$\geq \frac{\varepsilon}{t(1+\varepsilon)+\varepsilon} \# E_t$$

So $$\frac{1}{t([g_1,g_2]+k(\varepsilon))} \log \# \mathrm{Per}^{g_2}\left(t([g_1,g_2]+k(\varepsilon))\right)$$

$$\geq \left([g_1,g_2]+k(\varepsilon)\right)^{-1}\left[\frac{1}{t}\log \# E_t + \frac{1}{t}\left(\log\varepsilon - 2\log t - 2\log(4\varepsilon)\right)\right]$$

Taking limits as $t \to \infty$, we obtain

$$h^*(g_2) \geq h(\mu_L^{g_1}) \Big/ [g_1,g_2]+k(\varepsilon)$$

Since ε was arbitrary and $k(\varepsilon) \to 0$ as $\varepsilon \to 0$ we're done ■

In conclusion, (M,g) variable neg. curv. surface

$$h(\mu_L^{g}) \leq \underbrace{\left(\frac{-2\pi\chi(m)}{vg}\right)^{1/2}}_{\text{“}\, h^*(g_0)}$$

with equality iff constant neg. curv.

'88 Four applications of conformal equivalence in ... ETDS

Math 7222 Lecture 30

Last time:

In conclusion, (M, g) variable neg. curv. surface

$$h(\mu_L^g) \le \underbrace{\left(\frac{-2\pi\chi(M)}{V^g}\right)^{1/2}}_{= h^*(g_0)}$$

with equality iff constant neg. curv.

Today:

Theorem (M, g) closed Riem. surface $g_0 = \rho g$ where g_0 has constant $-$ve curvature (always happens when genus ≥ 2 by uniformization)

Assume $V^{g_0} = V^g$.

Then $h_{top}(g) \geq h_{top}(g_0) = \left(\frac{-2\pi\chi(M)}{V^g}\right)^{1/2}$

with equality iff $\rho \equiv 1$

i.e. $g \Rightarrow h_{top}(g)$ is minimized at constant neg. curvature

Consequence: For surfaces. Among neg. curv. metrics with fixed area, $h(\mu_L^g) = h_{top}(g)$ iff g has constant neg. curvature

In particular, $\mu_{BM}^g = \mu_L^g$ iff g has constant negative curvature.

Proof: It suffices to show $h_{top}(g) \geq h_{top}(g_0)$ where $g_0 = \rho g$, $V^{g_0} = V^{g}$, g_0 constant neg. curv.

For large T, consider $Per^{g_0}(T)$

We know $\frac{1}{T} \log \# Per^{g_0}(T) \to h_{top}(g_0)$

We also know that $\mu_L^{g_0}$ is the unique MME for $(f_t^{g_0})$. For $\varphi \in C(SM)$ let $\varepsilon > 0$ and define

$$Per^{g_0}(t, \varepsilon, \varphi) = \left\{ \gamma \in Per^{g}(t) : \left| \frac{\int \varphi \, d\mu_\gamma}{L^{g_0}(\gamma)} - \int \varphi \, d\mu_L^{g_0} \right| < \varepsilon \right\}$$

$= \int_0^{L(\gamma)} \varphi(f_s^{g_0} \dot\gamma(0)) \, ds$

Claim: $$\frac{\# Per^{g_0}(t, \varepsilon, \varphi)}{\# Per(t)} \to 1 \text{ as } t \to \infty$$

Sketch proof: If not, there are "enough" (i.e. $\geq \varepsilon_0 \# Per(t)$) γ's not in $Per^{g_0}(t, \varepsilon, \varphi)$ to construct an MME, m. By construction $\int \varphi \, dm \neq \int \varphi \, d\mu_L$. By uniqueness of the MME, this is a contradiction ∎

Now assume T is large enough so that

$$\# Per(T, \varepsilon, \rho^{-1/2}) > (1 - \varepsilon) \# Per(T)$$

(where $g_0 = \rho g$)

Note that $\int_{S^{g_0}M} \rho^{-\frac{1}{2}} d\mu_L^{g_0} = \int_M \rho^{-1/2} dm^{g_0}$

$= \int_M \rho^{\frac{1}{2}} dm^g$

Lecture 28 Lemma 1 $\geq = [g, \rho g] = [g, g_0]$

Note that $L^g(\gamma) = \int_0^{L^{g_0}(\gamma)} \| f_s^{g_0} \dot{\gamma}(0) \|_g \, ds$

$= \int \sqrt{\rho^{-1} g_0(\cdot,\cdot)} \, ds$

$= \int_0^{l^{g_0}(\gamma)} \rho^{-1/2} \, ds$

We have $L^g(\gamma) = L^{g_0}(\gamma) \cdot \left(\begin{matrix}\text{average of } \rho^{-1/2} \\ \text{along } \gamma\end{matrix} \right)$

For $\gamma \in Per^{g_0}(T, \varepsilon, \rho^{-\frac{1}{2}})$

then $L^g(\gamma) \leq T([g, g_0] + \varepsilon)$

Replace each such γ by g-shortest curve in its homotopy class. We have shown that

$$\# Per^g(T([g, g_0] + \varepsilon)) \geq \# Per^{g_0}(T, \varepsilon, \rho^{-\frac{1}{2}})$$

$$\geq (1-\varepsilon) \# Per^{g_0}(t)$$

It follows that

$$h^*(g) \geq [g, g_0]^{-1} h^*(g_0)$$

Recall $[g, g_0] \leq 1$ with equality iff $g = g_0$

Thus $h^*(g) \geq h^*(g_0)$ with equality iff $\rho = 1$ ∎

Higher dimensions

Besson - Courtois - Gallot showed that $h_{top}(g)$ is minimized at constant neg. curvature in all dimensions (90's)
(actually at neg. curv. locally symmetric)

- $g \Rightarrow h(\mu_L^g)$ is a mystery in higher dimensions. Definitely not maximised at constant curvature (Flaminio)
- $g \Rightarrow h_{top}(g) - h(\mu_L^g)$??? . . .

Katok entropy conjecture says that $h_{top}(g) = h(\mu_L^g)$ iff g locally symmetric
Open and challenging!

[Foulon: Extension of Katok to 3D contact Anosov flows]